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Diffusions on symmetric spaces of type A III and random matrix theories for rectangular matrices

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Abstract. Isotropic diffusion processes on cosets $U(M+N)/U(M)\times U(N)$ and $U(M,N)/U(M)\times U(N)$ and their zero-curvature limit are studied from a unified viewpoint. As indicated in our previous works the projection of the Fokker–Planck equation onto the maximal commutative subgroup of these cosets can be described by using the radial part of the Laplace–Beltrami operator. By taking the zero-curvature limit, an integral which is a natural extension of the Itzykson–Zuber integral to the rectangular matrices is explicitly evaluated. The probability density function obtained from the diffusion on $U(M,N)/U(M)\times U(N)$ is studied in detail, which can be applied to the quantum transport problem. The explicit expressions for the probability density function in the metallic and insulating regimes are obtained. For the metallic regime the integral representation for the hypergeometric function is used and the results are exact. Furthermore, by using the orthogonal polynomial method n-point correlation functions are obtained exactly for arbitrary n.

1. Introduction

Non-Hermitian random matrices have been applied to many areas in physics. For example they play a crucial role in the study of disordered systems, such as one-dimensional quantum transport [1–3]. Related to the subject, non-Hermitian regular random matrices are studied in [4–6]. Recently, rectangular random matrices have also attracted the attention of theorists in several fields [7, 8]. As far as the authors know, rectangular random matrices are not well investigated compared with the regular ones.

In this article we study a stochastic process on the cosets $U(M+N)/U(M)\times U(N)$ and $U(M,N)/U(M)\times U(N)$, where $U(M)\times U(N)$ denotes the direct product of the unitary groups $U(M)$ and $U(N)$. These cosets are classified as Riemannian symmetric spaces of type A III [9]. Our model is constructed so that in the zero-curvature limit the process reduces to the usual Brownian motion in $2MN$-dimensional Euclidian space, or $N\times M$ complex matrices. The coset spaces with non-zero curvature seem somewhat difficult to deal with. We may, however, get along with these non-zero curvature spaces to obtain the Fokker–Planck equations projected onto the maximal commutative subgroup of the cosets. Consequently, it is clarified that we can use the technique developed in the study of the Riemannian symmetric space. The diffusion equations thus obtained can be mapped to imaginary-time Schrödinger equations and analysed exactly.

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The zero-curvature limit is worth an independent investigation. In [10], the integral

$$\int_{U(N)} \mathrm{d}\mu^N(U) \exp[\beta \operatorname{tr}(M_1 U M_2 U^\dagger)] \tag{1.1}$$

$$M_1, M_2 \in u(N) \tag{1.2}$$

μ^N : normalized two-sided invariant Haar measure on $U(N)$

is evaluated by introducing the isotropic diffusion process of Hermitian matrices. Here, tr stands for the trace of matrix and † means the Hermitian conjugate. We refer to the integral (1.1) as the Itzykson–Zuber integral. Besides its significance in the mathematical context, the Itzykson–Zuber integral has many important applications in physics. For example, an application to quantum chaotic systems is studied in [11]. It is also fully used in the lattice gauge theory [12]. We might think of extensions of the Itzykson–Zuber integral. We can evaluate the Itzykson–Zuber-type integral by taking the zero-curvature limit of the diffusion on the symmetric spaces of type A III. Evaluated from our model is an integral

$$\int_{U(M)} \mathrm{d}\mu^M(U) \int_{U(N)} \mathrm{d}\mu^N(V) \exp\left(\frac{1}{t}\operatorname{Re}\operatorname{tr}(VYUX^\dagger)\right) \tag{1.3}$$

$X, Y : N \times M$ complex matrices

which can be realized as an extension of the Itzykson–Zuber integral to rectangular matrices.

Further analyses are possible also for the curvatured versions. Since we are motivated by the extended random matrix theory which can be applied to the one-dimensional quantum transport we choose the diffusion on the non-compact version $U(M, N)/U(M) \times (N)$ as a starting point of our analysis. By using the integral representation of the hypergeometric function it is possible to express the projection of the probability density function to the maximal commutative subgroup for the diffusion in an exact form for the metallic regime. Through a standard procedure we can also obtain the explicit form of the probability density function in the insulating regime. We shall see that they have, in common, an interaction term among the coordinates $\{x_i\}$ of the maximal commutative subgroup,

$$\prod_{i,j}(x_i - x_j) \sinh a(x_i - x_j) \tag{1.4}$$

which does not differ from the well-investigated case, $M = N$. This implies that the properties of our model are completely the same as in the case $M = N$ if we use the Dyson-gas approach. We can, however, integrate the probability density function for arbitrary numbers of points and obtain the n-point correlation function by the method of orthogonal polynomial.

This paper is organized as follows. Section 2 deals with the isotropic diffusion on $U(M+N)/U(M)\times U(N)$. The isotropic diffusion on $U(M, N)/U(M)\times U(N)$ in section 3 is the non-compact version of section 2. An almost parallel argument is possible and therefore we do not write details of calculations. In section 4 we consider the zero-curvature limit and show that the Itzykson–Zuber-type integral is evaluated in the sequel. At the beginning of section 5 we briefly illustrate how the diffusion on $U(M, N)/U(M) \times U(N)$ is related to the quantum transport. In the rest of the section we analyse the probability density projected to the maximal commutative subgroup for this diffusion process. Section 6 is a summary of this article which also contains some discussions. Throughout the article, we assume $M \geqslant N$.

2. Diffusion on $U(M+N)/U(M)\times U(N)$

Let $\mathfrak{G}$ and $\mathfrak{K}$ denote respectively the Lie algebras of a Lie group G and its subgroup K. We call $\mathfrak{J}=\mathfrak{G}-\mathfrak{K}$ the standard complementary space of the pair (G,K) [13]. The time evolution of a matrix $\mathrm{e}^{W(t)}$ is determined by

$$\mathrm{e}^{R(\mathrm{d}t,t)}\mathrm{e}^{W(t)}\sim\mathrm{e}^{W(t+\mathrm{d}t)} \tag{2.1}$$

$$W(t),\,R(\mathrm{d}t,t)\in\mathfrak{J} \tag{2.2}$$

where $\sim$ means an equivalence relation,

$$\alpha\sim\beta\Longleftrightarrow\alpha^{-1}\beta\in K\qquad\text{for }\alpha,\beta\in G. \tag{2.3}$$

So, if we would analyse the model with a usual manipulation of matrices, it is necessary to determine B such that

$$\mathrm{e}^{R(\mathrm{d}t,t)}\mathrm{e}^{W(t)}\mathrm{e}^{B(\mathrm{d}t,t)}=\mathrm{e}^{W(t+\mathrm{d}t)} \tag{2.4}$$

$$W(t),\,R(\mathrm{d}t,t)\in\mathfrak{J}\qquad B(\mathrm{d}t,t)\in\mathfrak{K}. \tag{2.5}$$

For the stochastic process, W, R and B are random matrices. In this section we deal with the Brownian motion, or a time-dependent random matrix on a coset $U(M+N)/U(M)\times U(N)$. We set $G=U(M+N)$ and $K=U(M)\times U(N)$. The rank of the coset $U(M+N)/U(M)\times U(N)$ is $\min(M,N)$. Since $W,R\in\mathfrak{J}$, they are parametrized by using $N\times M$ matrices X and Q as

$$W=\mathrm{i}a\begin{pmatrix}0 & X^\dagger\\ X & 0\end{pmatrix} \tag{2.6}$$

$$R=\mathrm{i}a\begin{pmatrix}0 & Q^\dagger\\ Q & 0\end{pmatrix} \tag{2.7}$$

where a is a real constant and plays the role of a scaling parameter. We assume that

$$\lim_{\mathrm{d}t\to0}\langle Q_{ij}(\mathrm{d}t,t)Q^*_{kl}(\mathrm{d}t,t)\rangle/\mathrm{d}t=\delta_{ik}\delta_{jl} \tag{2.8}$$

$$\lim_{\mathrm{d}t\to0}[\text{all other moments of the elements of }Q]/\mathrm{d}t=0 \tag{2.9}$$

where the brackets $\langle\ \rangle$ stand for the ensemble averaging. The $(M+N)\times(M+N)$ matrix W can be decomposed as follows:

$$W=\mathrm{i}a\begin{pmatrix}U^\dagger & 0\\ 0 & V\end{pmatrix}\begin{pmatrix}0 & \Xi^T\\ \Xi & 0\end{pmatrix}\begin{pmatrix}U & 0\\ 0 & V^\dagger\end{pmatrix} \tag{2.10}$$

$$\Xi=\begin{pmatrix}x_1 & 0 & \ldots & 0 & \ldots & 0\\ 0 & x_2 & \ldots & 0 & \ldots & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots\\ 0 & 0 & \ldots & x_N & \ldots & 0\end{pmatrix} \tag{2.11}$$

where T means the transposition. Expression (2.10) indicates that the eigenvalues of W consist of N pure-imaginary complex conjugate pairs $\{\pm\mathrm{i}ax_i|1\leqslant i\leqslant N\}$ and $M-N$ zeros.

Let $A(G/K)$ denote regular functions on G/K. As has been indicated in [14], the Fokker–Planck equation of the stochastic process described above can be mapped to a differential equation

$$\frac{\partial}{\partial t}\Psi=a^2\mathrm{div}\,\mathrm{grad}\,\Psi=a^2\mathcal{L}\Psi\qquad\Psi\in A(G/K) \tag{2.12}$$

where $\mathcal{L}$ is the so-called Laplace–Beltrami operator on G/K. In the conventional random matrix theories the ensemble of whole matrix elements is projected onto the ensemble of eigenvalues. For the coset G/K where G is compact this procedure can be generalized to the projection onto a maximal torus subgroup $\hat{T}$ of G. We express the action of $\mathfrak{G}$ on $A(G/K)$ from the left- and right-hand sides respectively by

$$g \cdot \Psi \qquad \text{and} \qquad \Psi \cdot g \qquad (g \in \mathfrak{G}, \Psi \in A(G/K)). \tag{2.13}$$

We set

$$A(K\backslash G/K) = \{\Psi \in A(G/K) | k \cdot \Psi = \Psi \qquad k \in \mathfrak{K}\}. \tag{2.14}$$

Namely, $A(K\backslash G/K)$ is the space of functions in $A(G/K)$ invariant under the action of K from the left-hand side. There is an isomorphism between $A(K\backslash G/K)$ and $A(\hat{T})$ with the identification of elements which can be transformed to each other by its Weyl group. The restriction of (2.12) on $A(K\backslash G/K)$ introduces the radial part of the Laplace–Beltrami operator $\mathcal{L}_+^r$,

$$\frac{\partial}{\partial t} f = a^2 \mathcal{L}_+^r f \qquad f \in A(K\backslash G/K) \tag{2.15}$$

$$\mathcal{L}_+^r = \frac{1}{2a^2} \xi_+(x)^{-2} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \xi_+(x)^2 \frac{\partial}{\partial x_i} \tag{2.16}$$

$$\xi_+(x) = \prod_{i<j} \sin a(x_i - x_j) \sin a(x_i + x_j) \prod_i \sin^{1/2}(2ax_i) \sin^{(M-N)}(ax_i). \tag{2.17}$$

The correspondence between the differential equation (2.15) and the diffusion process is seen as follows. The Fokker–Planck equation for $\{x_i\}$ is mapped to (2.15) by relating the probability density function $P(\{x_i\}, t)$ to $f(\{x_i\}, t)$,

$$P(\{x_i\}, t) = (\xi_+(x))^2 f(\{x_i\}, t). \tag{2.18}$$

Equation (2.15) can also be transformed into an imaginary-time Schrödinger equation for $\psi = \xi_+ f$:

$$-\frac{\partial}{\partial t} \psi = H_+ \psi \tag{2.19}$$

$$H_+ = -\frac{1}{2} \sum \frac{\partial^2}{\partial x_i^2} + \frac{(M-N)^2 a^2}{2} \sum \frac{1}{\sin^2 ax_i} - \frac{a^2}{2} \sum \frac{1}{\sin^2 2ax_i} - \frac{a^2 N}{6}(3M^2 + N^2 - 1). \tag{2.20}$$

Since the Hamiltonian H_+ does not contain interaction terms between 'particles', the eigenfunction can be written as a product of eigenfunctions for the one-particle Hamiltonians,

$$h_+^i = -\frac{1}{2} \frac{\partial^2}{\partial x_i^2} + \frac{(M-N)^2 a^2}{2} \frac{1}{\sin^2 ax_i} - \frac{a^2}{2} \frac{1}{\sin^2 2ax_i} - \frac{a^2}{6}(3M^2 + N^2 - 1). \tag{2.21}$$

Let us determine the eigenfunctions $\tilde{\psi}$ of the equation

$$h_+^i \tilde{\psi}(x_i) = E \tilde{\psi}(x_i). \tag{2.22}$$

We define a function η_+ by

$$\eta_+(x_i) = \sin^{1/2}(2ax_i) \sin^{(M-N)}(ax_i). \tag{2.23}$$

In terms of $\tilde{f} = \eta_+^{-1} \tilde{\psi}$, (2.22) is transformed again to

$$-\frac{1}{2\eta_+(x_i)^2} \frac{\mathrm{d}}{\mathrm{d}x_i} \eta_+(x_i)^2 \frac{\mathrm{d}}{\mathrm{d}x_i} \tilde{f}(x_i) = (E + \tilde{c}) \tilde{f}(x_i) \tag{2.24}$$

where

$$\tilde{c} = \frac{a^2}{3}(N+1)(3M-N-2). \tag{2.25}$$

By changing the variable

$$z = \tfrac{1}{2}(1-\cos(2ax_i)) = \sin^2(ax_i) \tag{2.26}$$

(2.24) is rewritten into Jacobi's differential equation,

$$z(1-z)\frac{\mathrm{d}^2\tilde{f}}{\mathrm{d}z^2} + [(M-N+1)-(M-N+2)z]\frac{\mathrm{d}\tilde{f}}{\mathrm{d}z} + \frac{E+\tilde{c}}{2a^2}\tilde{f} = 0. \tag{2.27}$$

For the Jacobi's differential equation, readers can refer to [15] for instance. Thus, $\tilde{f}$ is expressed by the hypergeometric function as

$$\tilde{f}(x_i) = F(-\nu, M-N+1+\nu, M-N+1; \sin^2(ax_i)) \tag{2.28}$$

where ν is related to E by

$$E + \tilde{c} = 2a^2\nu(M-N+1+\nu) \tag{2.29}$$

or

$$E = 2a^2\left(\nu + \frac{M-N+1}{2}\right)^2 - \frac{a^2}{6}(M-N+1). \tag{2.30}$$

Equation (2.28) is the unique rational solution of equation (2.27) and we discard the other solution (which contains logarithmic functions). When $\nu \in \mathbb{N}$,

$$F(-\nu, M-N+1+\nu, M-N+1; \sin^2 ax_i) = \frac{\Gamma(\nu+1)\Gamma(M-N+1)}{\Gamma(M-N+\nu+1)} P_\nu^{(M-N,0)}(\cos 2ax_i) \tag{2.31}$$

is a polynomial of degree ν with respect to z which is known as Jacobi's polynomials. Indeed, $\nu \in \mathbb{N}$ must be satisfied by the requirement that $\tilde{f}(x_i)$ does not diverge for $x_i \in (0, 2\pi/a)$. The solution of (2.22) is

$$\tilde{\psi}(x_i) = \eta_+(x_i)F(-\nu, M-N+1+\nu, 1; \sin^2(ax_i)). \tag{2.32}$$

It must also be noted that the solution in terms of the hypergeometric functions is no more than a formal one. The infinite series (2.28) does not necessarily converge. So the expression

$$\tilde{f}(x_i) = \sin^{2(N-M)}(ax_i)F(1+\nu, N-M-\nu, 1; \cos^2(ax_i)) \tag{2.33}$$

which is obtained by the analytic continuation and converges for $x_i \in \mathbb{R}$ might be more appropriate.

3. Diffusion on $U(M,N)/U(M)\times U(N)$

For the non-compact, or the negative curvatured version $U(M,N)/U(M)\times U(N)$ of the symmetric space almost parallel argument as in the previous section is possible. Remember that $U(p,q)$ is the group of matrices g in $GL(p+q,\mathbb{C})$ which fulfil

$$gI_{p,q}g^\dagger = I_{p,q} \tag{3.1}$$

with

$$I_{p,q} = \begin{pmatrix} -1_p & 0 \\ 0 & 1_q \end{pmatrix}. \tag{3.2}$$

In the above, 1_n means the $n \times n$ unit matrix. Parametrizations of W and R are now

$$W = a \begin{pmatrix} 0 & X^\dagger \\ X & 0 \end{pmatrix} \tag{3.3}$$

$$R = a \begin{pmatrix} 0 & Q^\dagger \\ Q & 0 \end{pmatrix} \tag{3.4}$$

instead of (2.6) and (2.7). Here, a is a real constant. Again, we can map a differential equation

$$\frac{\partial}{\partial t} f = a^2 \mathcal{L}^r_- f \qquad f \in A(K\backslash G/K) \tag{3.5}$$

$$\mathcal{L}^r_- = \frac{1}{2a^2} \xi_-(x)^{-2} \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \xi_-(x)^2 \frac{\partial}{\partial x_i} \tag{3.6}$$

$$\xi_-(x) = \prod_{i<j} \sinh a(x_i - x_j) \sinh a(x_i + x_j) \prod_i \sinh^{1/2}(2ax_i) \sinh^{(M-N)}(ax_i) \tag{3.7}$$

to an imaginary-time Schrödinger equation

$$-\frac{\partial}{\partial t} \psi = H_- \psi \tag{3.8}$$

$$H_- = -\frac{1}{2} \sum \frac{\partial^2}{\partial x_i^2} + \frac{(M-N)^2 a^2}{2} \sum \frac{1}{\sinh^2 ax_i} - \frac{a^2}{2} \sum \frac{1}{\sinh^2 2ax_i} + \frac{a^2 N}{6}(3M^2 + N^2 - 1). \tag{3.9}$$

The one-particle Schrödinger equation

$$h_i^- \tilde{\psi}(x_i) = E \tilde{\psi}(x_i) \tag{3.10}$$

has solutions

$$\tilde{\psi}(x_i) = \eta_-(x_i) F(-\nu, M - N + 1 + \nu, M - N + 1; -\sinh^2(ax_i)) \tag{3.11}$$

where

$$\eta_-(x_i) = \sinh^{1/2}(2ax_i) \sinh^{(M-N)}(ax_i) \tag{3.12}$$

and ν and E are related by

$$E - \tilde{c} = -2a^2 \nu (M - N + 1 + \nu). \tag{3.13}$$

Eigenfunctions of (3.8) are written as products of one-particle eigenfunctions. We want to have a system of real wave functions which approach the plain waves for large x_i. An appropriate parametrization for ν is

$$\nu = -\frac{1}{2}(M - N + 1) + \frac{\mathrm{i}}{2} k \qquad k \in \mathbb{R}. \tag{3.14}$$

Then E is expressed as a function of k by

$$E = \epsilon(k) = \frac{a^2 k^2}{2} + \frac{a^2}{6}(3M^2 + N^2 - 1). \tag{3.15}$$

Let us determine the orthonormal basis. We set $\mu = M - N + 1$ and

$$F_k(z) = F\left(\frac{\mu - \mathrm{i}k}{2}, \frac{\mu + \mathrm{i}k}{2}, \mu, z\right). \tag{3.16}$$

Introducing the normalization constant $C(k)$ we denote a normalized one-particle wavefunction by

$$\tilde{\psi}_k(x) = C(k)\eta_-(x)F_k(-\sinh^2(ax)). \tag{3.17}$$

$C(k)$ is determined by the relation

$$C(k)C(k')(-1)^\mu \int_{-\infty}^0 \mathrm{d}z\, z^{\mu-1}F_k(z)F_{k'}(z) = \delta(k-k') + \delta(k+k'). \tag{3.18}$$

Noting that F_k is a solution of the differential equation,

$$\left[\frac{\mathrm{d}}{\mathrm{d}z}z^\mu(1-z)\frac{\mathrm{d}}{\mathrm{d}z}\right]F_k - \frac{\mu^2+k^2}{4}z^{\mu-1}F_k = 0 \tag{3.19}$$

it follows that

$$\frac{k^2-k'^2}{4}\int_{-\infty}^0 \mathrm{d}z\, z^{\mu-1}F_k(z)F_{k'}(z) = \left[z^\mu(1-z)\left(F_{k'}\frac{\mathrm{d}F_k}{\mathrm{d}z} - F_k\frac{\mathrm{d}F_{k'}}{\mathrm{d}z}\right)\right]_{-\infty}^0. \tag{3.20}$$

By using the transformation formula

$$\begin{aligned}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma)}F(\alpha,\beta,\gamma;z) &= \frac{\Gamma(\alpha)\Gamma(\beta-\alpha)}{\Gamma(\gamma-\alpha)}(-z)^{-\alpha}F\left(\alpha,\alpha-\gamma+1,\alpha-\beta+1;\frac{1}{z}\right)\\ &\quad+\frac{\Gamma(\beta)\Gamma(\alpha-\beta)}{\Gamma(\gamma-\alpha)}(-z)^{-\beta}F\left(\beta,\beta-\gamma+1,\beta-\alpha+1;\frac{1}{z}\right)\end{aligned} \tag{3.21}$$

we get the asymptotic form for $|z| \to \infty$,

$$F\left(\frac{\mu}{2}-\frac{\mathrm{i}k}{2},\frac{\mu}{2}+\frac{\mathrm{i}k}{2},\mu;z\right) \longrightarrow 2\mathrm{Re}\left[\frac{\Gamma(\mu)\Gamma(\mathrm{i}k)}{\Gamma(\frac{\mu+\mathrm{i}k}{2})^2}(-z)^{(-\mu+\mathrm{i}k)/2}\right]. \tag{3.22}$$

Combining the above results, we obtain

$$\begin{aligned}\int_{-\infty}^0 \mathrm{d}z\, z^{\mu-1}F_k(z)F_{k'}(z) &= (-1)^\mu\frac{4\Gamma(\mu)^2}{k^2-k'^2}\lim_{z\to-\infty}\\ &\quad\times\mathrm{Re}\left[\mathrm{i}(k-k')\left(\frac{\Gamma(\mathrm{i}k)\Gamma(\mathrm{i}k')}{[\Gamma(\frac{\mu+\mathrm{i}k}{2})\Gamma(\frac{\mu+\mathrm{i}k'}{2})]^2}\right)(-z)^{\mathrm{i}(k+k')/2}\right.\\ &\quad\left.-\mathrm{i}(k+k')\left(\frac{\Gamma(-\mathrm{i}k)\Gamma(\mathrm{i}k')}{[\Gamma(\frac{\mu-\mathrm{i}k}{2})\Gamma(\frac{\mu+\mathrm{i}k'}{2})]^2}\right)(-z)^{\mathrm{i}(-k+k')/2}\right].\end{aligned} \tag{3.23}$$

It is easy to prove the orthogonality from (3.23). Let us integrate (3.23) with respect to k from zero to $+\infty$;

$$\begin{aligned}\int_0^\infty \mathrm{d}k\int_{-\infty}^0 \mathrm{d}z\, z^{\mu-1}F_k(z)F_{k'}(z) &= 4(-1)^\mu\Gamma(\mu)^2\mathrm{i}\lim_{z\to-\infty}\\ &\quad\times\left[\int_{-\infty}^\infty \mathrm{d}k\,\frac{1}{k+k'}\left(\frac{\Gamma(\mathrm{i}k)\Gamma(\mathrm{i}k')}{[\Gamma(\frac{\mu+\mathrm{i}k}{2})\Gamma(\frac{\mu+\mathrm{i}k'}{2})]^2}\right)(-z)^{\mathrm{i}(k+k')/2}\right.\\ &\quad\left.-\int_{-\infty}^\infty \mathrm{d}k\,\frac{1}{k+k'}\left(\frac{\Gamma(-\mathrm{i}k)\Gamma(-\mathrm{i}k')}{[\Gamma(\frac{\mu-\mathrm{i}k}{2})\Gamma(\frac{\mu-\mathrm{i}k'}{2})]^2}\right)(-z)^{-\mathrm{i}(k+k')/2}\right].\end{aligned} \tag{3.24}$$

The integrants of the first and second integrals on the right-hand side of (3.24) do not have poles respectively in the lower-half plane and the upper-half plane. Thus, it can be integrated out;

$$= 8\pi(-1)^\mu\Gamma(\mu)^2\frac{\Gamma(-\mathrm{i}k')\Gamma(\mathrm{i}k')}{[\Gamma(\frac{\mu-\mathrm{i}k'}{2})\Gamma(\frac{\mu+\mathrm{i}k'}{2})]^2}. \tag{3.25}$$

This determines the normalization $C(k)$. That is, (3.18) is satisfied by setting

$$C(k) = \frac{1}{\sqrt{8\pi}} \frac{\Gamma(\frac{\mu+\mathrm{i}k}{2})\Gamma(\frac{\mu-\mathrm{i}k}{2})}{\Gamma(\mu)|\Gamma(\mathrm{i}k)|}. \tag{3.26}$$

4. Zero-curvature limit and the Itzykson–Zuber integral for rectangular matrices

Leaving considerations about the time-evolution of diffusion processes to later we focus in this section on the limit $a \to 0$. Whether we start from $U(M+N)/U(M) \times U(N)$ in section 2 or $U(M,N)/U(M) \times U(N)$ in section 3 does not matter. The results are completely the same. The Hamiltonian for the imaginary-time Schrödinger equation reduces to

$$\tilde{H} = \sum_i \tilde{h}^i_+ \tag{4.1}$$

where

$$\tilde{h}^i_+ = \lim_{a\to 0} h^i_+ = -\frac{1}{2}\frac{\partial^2}{\partial x_i^2} + \frac{(M-N)^2 - \frac{1}{4}}{2}\frac{1}{x_i^2}. \tag{4.2}$$

This limit corresponds to the stochastic process of $N \times M$ matrices,

$$\mathrm{d}\tilde{X}(t) = \tilde{Q}(t, \mathrm{d}t) \tag{4.3}$$

$$\lim_{\mathrm{d}t\to 0} \langle \tilde{Q}_{ij}(\mathrm{d}t, t)\tilde{Q}^*_{kl}(\mathrm{d}t, t)\rangle/\mathrm{d}t = \delta_{ik}\delta_{jl} \tag{4.4}$$

$$\lim_{\mathrm{d}t\to 0} [\text{all other moments of the elements of } \tilde{Q}]/\mathrm{d}t = 0. \tag{4.5}$$

That is, the procedure $a \to 0$ means the zero-curvature limit. We define ξ_0 by

$$\xi_0(x) = \prod_{i,j}(x_i^2 - x_j^2)\prod_i x_i^{M-N+\frac{1}{2}} \tag{4.6}$$

which is essentially the $a \to 0$ limit of (2.17). The eigenfunction of the one-particle Hamiltonian $\tilde{h}^i_+$ which does not diverge at $x = 0$ is written in terms of the Bessel function,

$$\psi_k = (kx)^{1/2} J_{M-N}(kx) \qquad k \in \mathbb{R} \tag{4.7}$$

upto the normalization constant. The corresponding eigenvalue is $k^2/2$. The one-particle (imaginary-time) Green function for the Hamiltonian $\tilde{h}^i_+$ is given by

$$g_0(x, y; t) = \frac{x^{1/2}y^{1/2}}{t}\exp\left(-\frac{x^2+y^2}{2t}\right) I_{M-N}\left(\frac{xy}{t}\right) \tag{4.8}$$

where $I_{M-N}(x)$ is the modified Bessel function. Let us denote by $G_0(x, y; t)$ the probability density function which satisfies

$$\frac{\partial}{\partial t} G_0(x, y; t) = \mathcal{L}^r_0 G_0(x, y; t) \tag{4.9}$$

$$\mathcal{L}^r_0 = \lim_{a\to 0}[a^2\mathcal{L}^r_+] = \frac{1}{2}\xi_0(x)^{-2}\sum_{i=1}^{N}\frac{\partial}{\partial x_i}\xi_0(x)^2\frac{\partial}{\partial x_i} \tag{4.10}$$

and the initial condition

$$G_0(x, y; 0) = \frac{1}{N!}\sum_{P\in\sigma_N}\prod_{i=1}^{N}\delta(x_i - y_{P_i}) \tag{4.11}$$

where σ_N is the symmetric group of degree N, i.e. the summation is over all the permutations of $\{1, 2, \ldots, N\}$. From (4.8) we can show that

$$G_0(x, y; t) = \frac{\xi_0(x)}{\xi_0(y)} \frac{1}{N!} \prod_i \frac{x_i^{1/2} y_i^{1/2}}{t} \exp\left(-\frac{x_i^2 + y_i^2}{2t}\right) \left\{\det\{I_{M-N}\left(\frac{x_k y_l}{t}\right)\right\}_{k,l}. \tag{4.12}$$

Since we are dealing with the standard $2NM$-dimensional Brownian motion (See (4.3), (4.4) and (4.5)), the probability density function $G_0(x, y, t)$ has another expression:

$$\begin{aligned}
\int \prod_i \mathrm{d}x_i\, G_0(x, y; t) &= (2\pi t)^{-MN} \int \prod_{i=1}^{N} \prod_{j=1}^{M} \mathrm{dRe}\ (X_{ij})\, \mathrm{dIm}\, (X_{ij}) \int \mathrm{d}\mu^{\bar{M}}\ (U') \int \mathrm{d}\mu^{N}\ (V') \\
&\quad \times \exp\left(-\frac{1}{2t} \operatorname{tr}(X - V'\Psi U')(X - V'\Psi U')^{\dagger}\right) \\
&= C(2\pi t)^{-MN} \int \prod_i \mathrm{d}x_i\, \xi_0(x)^2 \int \mathrm{d}\mu^{\bar{M}}(U) \mathrm{d}\mu^{\bar{M}}\ (U') \\
&\quad \times \int \mathrm{d}\mu^{N}\ (V) \mathrm{d}\mu^{N}\ (V') \exp\left(-\frac{1}{2t} \operatorname{tr}(V\Xi U - V'\Psi U')(V\Xi U - V'\Psi U')^{\dagger}\right) \\
&= C(2\pi t)^{-MN} \int \prod_i \mathrm{d}x_i\, \xi_0(x)^2 \exp\left(-\frac{x_i^2 + y_i^2}{2t}\right) \\
&\quad \times \int \mathrm{d}\mu^{\bar{M}}\ (U) \int \mathrm{d}\mu^{N}\ (V) \exp\left(\frac{1}{t} \mathrm{Re}\ \operatorname{tr}(V\Psi U\Xi^{\dagger})\right)
\end{aligned} \tag{4.13}$$

where Ψ is an $N \times M$ matrix whose elements are

$$\Psi = \begin{pmatrix} y_1 & 0 & \ldots & 0 & \ldots & 0 \\ 0 & y_2 & \ldots & 0 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \ldots & y_N & \ldots & 0 \end{pmatrix}. \tag{4.14}$$

The normalization constant C accompanies the change of variable of integration:

$$C^{-1} \int \prod_{i=1}^{N} \prod_{j=1}^{M} \mathrm{dRe}\ (X_{ij}) \mathrm{dIm}\ (X_{ij}) = \int \prod_i \mathrm{d}x_i\, \xi_0(x)^2 \int \mathrm{d}\mu^{\bar{M}}\ (U) \int \mathrm{d}\mu^{N}\ (V) \tag{4.15}$$

$$X = V\Xi U \tag{4.16}$$

and will be fixed later. We express by $\mu^{\bar{M}}$ the normalized Haar measure of

$$U(M)/Z_K(\hat{T}) \tag{4.17}$$

where

$$Z_K(\hat{T}) = U(M-N) \times (U(1))^N \tag{4.18}$$

is realized as the centralizer of $\hat{T}$ in K. In the last expression of (4.13) we can replace an integral

$$\int_{U(M)/Z_K(\hat{T})} \mathrm{d}\mu^{\bar{M}}\ (U) \tag{4.19}$$

by an integral over $U(M)$,

$$\int_{U(M)} \mathrm{d}\mu^{M}\ (U). \tag{4.20}$$

By comparing (4.12) with (4.13) we find that the Itzykson–Zuber intergral for rectangular matrices is given by

$$\int \mathrm{d}\mu^M(U)\int \mathrm{d}\mu^N(V)\exp\left(\frac{1}{t}\mathrm{Re}\ \mathrm{tr}(V\Psi U\Xi^\dagger)\right)$$
$$= \frac{(2\pi)^{MN}}{CN!}t^{MN-N}\prod_i (x_i y_i)^{\frac{1}{2}}\frac{\det\{I_{M-N}(\frac{x_k y_l}{t})\}_{k,l}}{\xi_0(x)\xi_0(y)}. \tag{4.21}$$

The constant C is determined as follows. We set $\exp(-\mathrm{tr}(XX^\dagger)/2)$ as the integrand of (4.15). The left-hand-side is readily integrated and we obtain an equality,

$$(2\pi)^{MN}C^{-1} = \int_{-\infty}^{\infty}\prod_{i=1}^{N}\mathrm{d}x_i\ \xi_0(x)^2\exp\left(-\sum_i\frac{x_i^2}{2}\right). \tag{4.22}$$

The right-hand side of (4.22) is evaluated by using the Selberg's integral [16] to be

$$= 2^{-N}\int_0^\infty\prod_i \mathrm{d}y_i\ y_i^{M-N}\prod_{i<j}(y_i-y_j)^2\exp\left(-\sum_i\frac{y_i}{2}\right)$$
$$= 2^{N(M-N+1)+N(N-1)-N}\int_0^\infty\prod_i \mathrm{d}y_i\ y_i^{M-N}\prod_{i<j}(y_i-y_j)^2\exp\left(-\sum_i y_i\right)$$
$$= 2^{N(M-1)}\prod_{j=0}^{N}\frac{\Gamma(2+j)\Gamma(M-N+1+j)}{\Gamma(2)}$$
$$= 2^{N(M-1)}\prod_{j=0}^{N}(1+j)!(M-N+j)!. \tag{4.23}$$

Thus, the complete form of the Itzykson–Zuber integral for rectangular matrices is determined:

$$\int \mathrm{d}\mu^M(U)\int \mathrm{d}\mu^N(V)\exp\left(\frac{1}{t}\mathrm{Re}\ \mathrm{tr}(V\Psi U\Xi^\dagger)\right) = \frac{2^{N(M-1)}}{N!}t^{MN-N}$$
$$\times\prod_{j=0}^{N}(1+j)!(M-N+j)!\prod_i (x_i y_i)^{\frac{1}{2}}\frac{\det\{I_{M-N}(\frac{x_k y_l}{t})\}_{k,l}}{\xi_0(x)\xi_0(y)}. \tag{4.24}$$

5. Physical applications

5.1. Quantum transport; symmetries of transfer matrices

We consider an ensemble of transfer matrices which relate incoming and outgoing fluxes on both sides of a quasi-one-dimensional disordered mesoscopic system (figure 1). We assume that there are M right-moving channels and N left-moving channels on both sides of the system. not We write an nth state of right- (left-) moving channel in region A as $|n, A_+\rangle$ ($|n, A_-\rangle$). We decompose the wavefunctions in the regions I and III as

$$\Psi_{\mathrm{I}}(x,\sigma_z) = \sum_{n=1}^{M}a_{\mathrm{I}}^n|n;\mathrm{I}_+\rangle + \sum_{n=1}^{N}b_{\mathrm{I}}^n|n;\mathrm{I}_-\rangle \tag{5.1}$$

and

$$\Psi_{\mathrm{III}}(x,\sigma_z) = \sum_{n=1}^{M}a_{\mathrm{III}}^n|n;\mathrm{III}_+\rangle + \sum_{n=1}^{N}b_{\mathrm{III}}^n|n;\mathrm{III}_-\rangle. \tag{5.2}$$

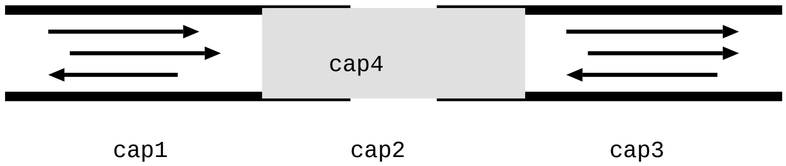

Figure 1. A quasi-one-dimensional mesoscopic system. Region II is disorderd while regions I and III do not contain disorders.

The states $|n : \mathrm{I}_\pm\rangle$ and $|n : \mathrm{III}_\pm\rangle$ are vectors with normalized moments;

$$\begin{aligned} \langle n : \mathrm{I}_\pm| - \mathrm{i}\partial_x |n : \mathrm{I}_\pm\rangle &= \pm 1 \\ \langle n : \mathrm{III}_\pm| - \mathrm{i}\partial_x |n : \mathrm{III}_\pm\rangle &= \pm 1. \end{aligned} \tag{5.3}$$

We denote by $\boldsymbol{a}_\mathrm{I}$ and $\boldsymbol{a}_\mathrm{III}$, respectively, the M-component vectors a^n_I and $a^n_\mathrm{III}(1 \leqslant n \leqslant M)$. In the same way $\boldsymbol{b}_\mathrm{I}$ and $\boldsymbol{b}_\mathrm{III}$ are understood as the N-component vectors. Then the transfer matrix R is an $(M+N)\times(M+N)$ matrix defined by

$$\begin{pmatrix} \boldsymbol{a}_\mathrm{III} \\ \boldsymbol{b}_\mathrm{III} \end{pmatrix} = R \begin{pmatrix} \boldsymbol{a}_\mathrm{I} \\ \boldsymbol{b}_\mathrm{I} \end{pmatrix}. \tag{5.4}$$

It is convenient to define

$$\boldsymbol{C}_\mathrm{I} = \begin{pmatrix} \boldsymbol{a}_\mathrm{I} \\ \boldsymbol{b}_\mathrm{I} \end{pmatrix} \qquad \boldsymbol{C}_\mathrm{III} = \begin{pmatrix} \boldsymbol{a}_\mathrm{III} \\ \boldsymbol{b}_\mathrm{III} \end{pmatrix}. \tag{5.5}$$

We impose the flux conservation constraint on R. In terms of

$$\sigma_z = \begin{pmatrix} 1_M & 0 \\ 0 & -1_N \end{pmatrix} \tag{5.6}$$

this constraint is represented as

$$\boldsymbol{C}_\mathrm{I}^\dagger \sigma_z \boldsymbol{C}_\mathrm{I} = \boldsymbol{C}_\mathrm{III}^\dagger \sigma_z \boldsymbol{C}_\mathrm{III}. \tag{5.7}$$

This relation is equivalent to

$$\sigma_z = R^\dagger \sigma_z R \tag{5.8}$$

which means only [9] that the transfer matrix R belongs to $U(M,N)$,

$$R \in U(M,N). \tag{5.9}$$

In the case $M = N$, a classification by the following three universality classes is useful [17];

- $\beta = 1$: invariant under time-reversal (integer spin);
- $\beta = 2$: no time-reversal invariance;
- $\beta = 4$: invariant under time-reversal (half-odd spin).

We assume, however, that $M \neq N$. This is not an unnatural assumption. When $M \neq N$ there cannot exist time-reversal symmetry. One of the examples of this situation is the chiral Luttinger liquid which appears as the edge state of the fractional quantum Hall system. So we shall concentrate on the case $\beta = 2$. Let us consider the change of the transfer matrix due to an infinitesimal increase of the length 't' of the region II. Since the region II is disorderd, it is natural to consider that the transfer matrix is multiplied by an infinitesimal random matrix which belongs to $U(M,N)$. We do not know about the bases used for the decomposition of the wavefunction for the regions I and Ⅲ. Therefore, we identify two transfer matrices which can be transformed to each other by the unitary rotations of the basis,

$$\begin{aligned} a_{\mathrm{III}} &\longrightarrow u a_{\mathrm{III}} \qquad (u \in U(M)) \\ b_{\mathrm{III}} &\longrightarrow v b_{\mathrm{III}} \qquad (v \in U(N)). \end{aligned} \tag{5.10}$$

Then the isotropic diffusion of the transfer matrix is described by the model discussed in section 3.

5.2. Time evolution for $U(M, N)/U(M) \times U(N)$

Let us further discuss the diffusion. We choose $U(M, N)/U(M) \times U(N)$ as an example. The probability density function projected to for the maximal commutative subgroup is expressed in almost the same way as (4.12),

$$G(x, y; t) = \frac{\xi_-(x)}{\xi_-(y)} \frac{1}{N!} \det\{g(x_i, y_j; t)\}_{i,j}. \tag{5.11}$$

Here $g(x_i, y_j; t)$ is the one-particle Green function defined by

$$g(x_i, y_j; t) = \int \mathrm{d}k \; \exp[-\epsilon(k)t] \tilde{\psi}_k(x_i) \tilde{\psi}_k(y_j). \tag{5.12}$$

From (5.11) and (5.12), we have

$$\begin{aligned} G(x, y; t) &= \frac{\xi_-(x)}{\xi_-(y)} \frac{1}{N!} \left[\prod_{i=1}^{N} \int \mathrm{d}k_i \; \exp[-\epsilon(k_i)t] \tilde{\psi}_k(x_i) \right] \det\{\tilde{\psi}_{k_i}(y_j)\}_{i,j} \\ &= \prod_{i<j} (\sinh^2 x_i - \sinh^2 x_j) \left[\prod_{i=1}^{N} \int \; \mathrm{d}k_i \right] \\ &\quad \times \prod_{i=1}^{N} \left[\eta_-^2(x_i) C(k_i)^2 \exp[-\epsilon(k_i)t] F\left(\frac{\mu - \mathrm{i}k_i}{2}, \frac{\mu + \mathrm{i}k_i}{2}, \mu; -\sinh^2(ax_i) \right) \right] \\ &\quad \times \frac{\det\{F(\frac{\mu - \mathrm{i}k_i}{2}, \frac{\mu + \mathrm{i}k_i}{2}, \mu; -\sinh^2(ay_j))\}_{i,j}}{\prod_{i<j} (\sinh^2 y_i - \sinh^2 y_j)} \end{aligned} \tag{5.13}$$

where $\mu = M - N + 1$. We focus on the limit $y_i \to 0$ for all i. If we note that

$$F\left(\frac{\mu - \mathrm{i}k_i}{2}, \frac{\mu + \mathrm{i}k_i}{2}, \mu; -\sinh^2(ay_j) \right) = 1 + \sum_{m=1}^{\infty} \frac{\prod_{n=1}^{m} [(\mu + 2n)^2 + k_i^2]}{2^{2m} (\mu)_m} \frac{(-\sinh^2(ay_i))^m}{m!} \tag{5.14}$$

we can readily show that

$$\left[\prod_i \lim_{y_i \to 0} \right] \frac{\det\{F(\frac{\mu - \mathrm{i}k_i}{2}, \frac{\mu + \mathrm{i}k_i}{2}, \mu; -\sinh^2(ay_j))\}_{i,j}}{\prod_{i<j} (\sinh^2 y_i - \sinh^2 y_j)} = \text{constant} \times \det\{k_i^{2(j-1)}\}_{i,j}. \tag{5.15}$$

Thus, from (5.13) and (5.15), we obtain

$$\begin{aligned} G(x, 0; t) &= \text{constant} \times \prod_{i<j} (\sinh^2 x_i - \sinh^2 x_j) \left[\prod_{i=1}^{N} \eta_-^2(x_i) \right] \\ &\quad \times \det \left\{ \int \mathrm{d}k_i \; k_i^{2(j-1)} C(k_i)^2 \exp[-\epsilon(k_i)t] \right. \\ &\quad \left. \times F\left(\frac{\mu - \mathrm{i}k_i}{2}, \frac{\mu + \mathrm{i}k_i}{2}, \mu; -\sinh^2(ax_i) \right) \right\}_{i,j}. \end{aligned} \tag{5.16}$$

5.3. Metallic regime

5.3.1. even $M - N$. When $M - N$ is a non-negative even integer, (5.16) is rewritten as (see (A.6))

$$G(x, 0; t) = c_l \prod_{i<j} (\sinh^2 x_i - \sinh^2 x_j) \left[\prod_{i=1}^{N} \eta_-^2(x_i) \right] \exp\left[-\frac{a^2}{6} N(3M^2 + N^2 - 1)t \right]$$
$$\times \det\left\{ \int_0^\infty \mathrm{d}k_i k_i^{2j-1} \tanh(\pi k_i/2) \prod_{n=1}^{L} [k_i^2 + (2n-1)^2]^2 \exp\left[-\frac{a^2 t k_i^2}{2} \right] \right.$$
$$\left. \times F\left(\frac{\mu - \mathrm{i}k_i}{2}, \frac{\mu + \mathrm{i}k_i}{2}, \mu; -\sinh^2(ax_i) \right) \right\}_{i,j} \tag{5.17}$$

where c_l is a constant and $L = (M - N)/2$. For brevity we define functions $A_j(j = 1, 2, \ldots, N)$ by

$$A_j(x_i) = \int_0^\infty \mathrm{d}k\, k^{2j-1} \tanh(\pi k/2) \prod_{n=1}^{L} [k^2 + (2n-1)^2]^2 \exp\left[-\frac{a^2 t k^2}{2} \right]$$
$$\times F\left(\frac{\mu - \mathrm{i}k}{2}, \frac{\mu + \mathrm{i}k}{2}, \mu; -\sinh^2(ax_i) \right)$$
$$= \int_{-\infty}^{\infty} \mathrm{d}k\, k^{2j-1} \frac{1}{2\cosh(\pi k/2)} \prod_{n=1}^{L} [k^2 + (2n-1)^2]^2$$
$$\times \exp\left[-\frac{a^2 t k^2}{2} + \pi k/2 \right] F\left(\frac{\mu - \mathrm{i}k}{2}, \frac{\mu + \mathrm{i}k}{2}, \mu; -\sinh^2(ax_i) \right). \tag{5.18}$$

The integral representation for the hypergeometric function

$$F(\alpha, \beta, \gamma; z) = \frac{\Gamma(\gamma)}{\Gamma(\gamma - \beta)\Gamma(\beta)} \int_0^1 s^{\beta-1} (1-s)^{\gamma-\beta-1} (1-sz)^{-\alpha}\, \mathrm{d}s \tag{5.19}$$

is useful here; by setting

$$b(x, s) = \ln\left[\frac{s(1 + s\sinh^2 ax)}{1-s} \right] \tag{5.20}$$

we have

$$F\left(\frac{\mu - \mathrm{i}k}{2}, \frac{\mu + \mathrm{i}k}{2}, \mu; -\sinh^2(ax) \right) = \frac{2^{2L-1}\Gamma(\mu)}{\pi} \cosh(\pi k/2) \prod_{n=1}^{L} [k^2 + (2n-1)^2]^{-1}$$
$$\times \int_0^1 \mathrm{d}s [s(1-s)]^{\mu/2-1} (1 + s\sinh^2 ax)^{-\mu/2} \exp\left[\frac{\mathrm{i}kb(x,s)}{2} \right]. \tag{5.21}$$

Use of (5.21) in (5.18) yields

$$A_j(x_i) = \frac{2^{2L-2}\Gamma(\mu)}{\pi} \int_0^1 \mathrm{d}s [s(1-s)]^{\mu/2-1} (1 + s\sinh^2 ax_i)^{-\mu/2} \int_{-\infty}^{\infty} \mathrm{d}k\, k^{2j-1}$$
$$\times \prod_{n=1}^{L} [k^2 + (2n-1)^2] \exp\left[\frac{-a^2 t k^2 + \mathrm{i}kb(x_i, s) + \pi k}{2} \right]. \tag{5.22}$$

Another useful formula is

$$\int_{-\infty}^{\infty} \mathrm{d}k\, k^m \exp\left[\frac{-a^2tk^2 + \mathrm{i}kb(x_i, s) + \pi k}{2}\right] = 2\sqrt{\pi}\mathrm{i}^m(2a^2t)^{-(m+1)/2} \times \exp\left[-\frac{1}{8a^2t}(b(x_i, s) - \mathrm{i}\pi)^2\right] H_m\left(\frac{b - \mathrm{i}\pi}{\sqrt{8a^2t}}\right) \tag{5.23}$$

where $H_m(x)$ is the Hermite polynomial. Substituting

$$\prod_{n=0}^{L}[k^2 + (2n-1)^2] = \sum_{n=1}^{L} q_n k^{2n} \tag{5.24}$$

and (5.23) into (5.22), we obtain

$$A_j(x_i) = \mathrm{i}\frac{2^{2L-1}\Gamma(\mu)}{\sqrt{\pi}} \sum_{n=0}^{L} q_n(-2a^2t)^{-(j+n)} \int_0^1 \mathrm{d}s\, H_{2(j+n)-1}\left(\frac{b - \mathrm{i}\pi}{\sqrt{8a^2t}}\right)[s(1-s)]^{\mu/2-1} \times (1 + s\sinh^2 ax_i)^{-\mu/2} b(x_i, s) 2a^2t \exp\left[-\frac{1}{8a^2t}(b(x_i, s) - \mathrm{i}\pi)^2\right]. \tag{5.25}$$

Let us consider the metallic regime where $t \ll N$. In this case the dominant contribution comes from leading-order terms with respect to t^{-1};

$$A_j(x_i) \approx \mathrm{i}\frac{2^{2L-1}\Gamma(\mu)}{\sqrt{\pi}}(-2a^2t)^{-(j+L)} \int_0^1 \mathrm{d}s \left(\frac{b - \mathrm{i}\pi}{\sqrt{2a^2t}}\right)^{2(j+L)-1} [s(1-s)]^{\mu/2-1} \times (1 + s\sinh^2 ax_i)^{-\mu/2} b(x_i, s) 2a^2t \exp\left[-\frac{1}{8a^2t}(b(x_i, s) - \mathrm{i}\pi)^2\right]. \tag{5.26}$$

We may evaluate the integral

$$\int_0^1 \mathrm{d}s\, (b(x_i, s) - \mathrm{i}\pi)^l [s(1-s)]^{\mu/2-1}(1 + s\sinh^2 ax_i)^{-\mu/2} b(x_i, s) 2a^2t \times \exp\left[-\frac{1}{8a^2t}(b(x_i, s) - \mathrm{i}\pi)^2\right] \tag{5.27}$$

by using the method of steepest descent. The exponential term of (5.27) has a saddle point at $s = 1 + \coth ax_i$, where $\mathrm{e}^b = -\mathrm{e}^{2ax_i}$. We can deform the contour continuously to that which goes through this saddle point. Consequently,

$$\int_0^1 \mathrm{d}s\, (b(x_i, s) - \mathrm{i}\pi)^l [s(1-s)]^{\mu/2-1}(1 + s\sinh^2 ax_i)^{-\mu/2} \exp\left[-\frac{1}{8a^2t}(b(x_i, s) - \mathrm{i}\pi)^2\right]$$
$$\approx \mathrm{i}(2ax_i)^l \mathrm{i}^{-\mu}(\sinh ax_i)^{-\mu+2}(\cosh ax_i)^{-1}\mathrm{e}^{-ax_i} \exp\left[-\frac{1}{2t}x_i^2\right]\sqrt{\frac{2at\pi\cosh ax_i}{x_i \mathrm{e}^{-2ax_i}\sinh^3 ax_i}}$$
$$= \text{constant} \times a^{l+\frac{1}{2}}\sqrt{x_i}x_i^{l-1}\sinh^{-\mu+1} ax_i \sinh^{-1/2} 2ax_i \exp\left[-\frac{1}{2t}x_i^2\right]. \tag{5.28}$$

Now (5.17) is expressed as

$$G(x, 0; t) = \text{constant} \times a^{N(M+\frac{1}{2})} \prod_{i<j}(\sinh^2 x_i - \sinh^2 x_j)(x_i^2 - x_j^2) \prod_{i=1}^{N} x_i^{M-N+\frac{1}{2}} \eta_-(x_i) \times \exp\left[-\frac{N}{2t}x_i^2 - \frac{a^2}{6}N(3M^2 + N^2 - 1)t\right]. \tag{5.29}$$

When $M - N = 0$ (5.29) reproduces the results in [3].

5.3.2. Odd $M - N$. For a positive odd $M - N$, we set $\mu = M - N + 1 = 2L'$. Then (5.16) is rewritten as (A.6)

$$G(x, 0; t) = \text{constant} \times \prod_{i<j} (\sinh^2 x_i - \sinh^2 x_j) \left[\prod_{i=1}^{N} \eta_-^2(x_i) \right] \times \exp\left[-\frac{a^2}{6} N(3M^2 + N^2 - 1)t \right] \det\{A'_j(x_i)\}_{i,j} \tag{5.30}$$

where $A'_j(x_i)$ is defined by

$$A'_j(x_i) = \int_0^\infty \mathrm{d}k_i\, k_i^{2j-3} \coth(\pi k_i/2) \prod_{n=1}^{L'} [k_i^2 + 4(n-1)^2]^2 \exp\left[-\frac{a^2 t k_i^2}{2} \right] \times F\left(\frac{\mu - \mathrm{i}k_i}{2}, \frac{\mu + \mathrm{i}k_i}{2}, \mu; -\sinh^2(ax_i) \right). \tag{5.31}$$

The integral representation for the hypergeometric function in (5.31) is

$$F\left(\frac{\mu - \mathrm{i}k}{2}, \frac{\mu + \mathrm{i}k}{2}, \mu; -\sinh^2(ax) \right) = \frac{2^{2L'-1}\Gamma(\mu)}{\pi} k \sinh(\pi k/2) \prod_{n=1}^{L'} [k^2 + 4(n-1)^2]^{-1} \times \int_0^1 \mathrm{d}s [s(1-s)]^{\mu/2-1} (1 + s\sinh^2 ax)^{-\mu/2} \exp\left[\frac{\mathrm{i}kb(x, s)}{2} \right] \tag{5.32}$$

where the function $b(x, s)$ is defined in (5.20). From (5.31) and (5.32), we have

$$A'_j(x_i) = \frac{2^{2L'-2}\Gamma(\mu)}{\pi} \int_0^1 \mathrm{d}s\, [s(1-s)]^{\mu/2-1} (1 + s\sinh^2 ax_i)^{-\mu/2} \int_{-\infty}^{\infty} \mathrm{d}k\, k^{2(j-1)} \times \prod_{n=1}^{L'} [k^2 + 4(n-1)^2] \exp\left[\frac{-a^2 t k^2 + \mathrm{i}kb(x_i, s) + \pi k}{2} \right]. \tag{5.33}$$

The asymptotic behaviour of A'_j in the metallic regime is determined in the same way as in section 5.3.1;

$$A'_j(x_i) \approx \mathrm{i} \frac{2^{2L'-1}\Gamma(\mu)}{\sqrt{\pi}} (-2a^2 t)^{\frac{1}{2}-j-L} \int_0^1 \mathrm{d}s \left(\frac{b - \mathrm{i}\pi}{\sqrt{2a^2 t}} \right)^{2(j+L-1)} [s(1-s)]^{\mu/2-1} \times (1 + s\sinh^2 ax_i)^{-\mu/2} b(x_i, s) 2a^2 t \exp\left[-\frac{1}{8a^2 t} (b(x_i, s) - \mathrm{i}\pi)^2 \right]. \tag{5.34}$$

Using (5.28) we arrive at the conclusion that (5.29) is also valid when $M - N$ is a positive odd integer. Thus, the Green function in the metallic regime for arbitrary M and N is obtained. Note that this result is exact in the limit $t \to 0$.

5.4. Insulating regime

In the large t limit, the dominant contribution to the probability density function comes from the range $x_k \gg 1$. In this range we can use the asymptotic form (3.22),

$$\int \mathrm{d}k\, k^{2(j-1)} C(k)^2 \exp[-\epsilon(k)t] F\left(\frac{\mu - \mathrm{i}k}{2}, \frac{\mu + \mathrm{i}k}{2}, \mu; -\sinh^2(ax_i) \right) \approx \frac{1}{4\pi\Gamma(\mu)^2} \exp\left[-\frac{a^2}{6} N(3M^2 + N^2 - 1)t \right]$$

$$\times \mathrm{Re}\left[\int_0^\infty \frac{\Gamma(\frac{\mu-\mathrm{i}k}{2})^2}{\Gamma(-\mathrm{i}k)} k^{2(j-1)} (\sinh ax_i)^{(-\mu+\mathrm{i}k)/2} \exp\left[-\frac{a^2tk^2}{2}\right]\right]$$
$$\approx \frac{(-1)^{j-1}(2a^2t)^{-j}}{2at\sqrt{\pi}\Gamma(\mu)^2\sinh^\mu(ax_i)} \exp\left[-\frac{a^2}{6}N(3M^2+N^2-1)t\right] x_i H_{2(j-1)}\left(\frac{x_i}{\sqrt{8t}}\right). \tag{5.35}$$

Inserting (5.35) in (5.16) we obtain

$$G(x,0;t) = \text{constant} \times \prod_{i<j}(\sinh^2 x_i - \sinh^2 x_j)(x_i^2 - x_j^2) \prod_{i=1}^N x_i \sinh^{M-N-1}(x_i)\sinh(2ax_i)$$
$$\times \exp\left[-\frac{N}{2t}x_i^2 - \frac{a^2}{6}N(3M^2+N^2-1)t\right]. \tag{5.36}$$

5.5. Orthogonal polynomial method

The aim of this section is the exact expression of the n-point correlation function,

$$R_n(z_1,\ldots,z_n;t) = \frac{N!}{(N-n)!}\int_{-\infty}^0 \mathrm{d}z_{n+1}\cdots\int_{-\infty}^0 \mathrm{d}z_N\, \hat{G}(z;t) \tag{5.37}$$

where

$$\hat{G}(z;t) = G(z,0;t). \tag{5.38}$$

This is achieved by the orthogonal polynomial method. [16, 18]. Let us set

$$z_i = -\sinh^2 ax_i \qquad (1 \leqslant i \leqslant N). \tag{5.39}$$

Taking the Jacobian of the transformation into account the probability density (5.16) is rewritten in terms of $\{z_i\}$ as

$$\hat{G}(z,t) = \text{constant} \times \prod_{i<j}(z_i - z_j)\prod_i z_i^{M-N} \det$$
$$\times\left\{\int \mathrm{d}k_i\, k_i^{2(j-1)} C(k_i)^2 \exp[-\epsilon(k_i)t] F\left(\frac{\mu-\mathrm{i}k_i}{2}, \frac{\mu+\mathrm{i}k_i}{2}, \mu; z_i\right)\right\}_{i,j}$$
$$= \text{constant} \times \prod_i z_i^{M-N} \det\{P_{i-1}^{(M-N,0)}(z_j)\}_{i,j} \det$$
$$\times\left\{\int \mathrm{d}k_i\, k_i^{2(j-1)} C(k_i)^2 \exp[-\epsilon(k_i)t] F\left(\frac{\mu-\mathrm{i}k_i}{2}, \frac{\mu+\mathrm{i}k_i}{2}, \mu; z_i\right)\right\}_{i,j}. \tag{5.40}$$

Following [18] we set

$$a_{mn}(t) = \int_{-\infty}^0 \mathrm{d}z\, z^{M-N} P_{m-1}^{(M-N,0)}(z) \int_0^\infty \mathrm{d}k\, k^{2(n-1)} C(k)^2 \exp[-\epsilon(k)t]$$
$$\times F\left(\frac{\mu-\mathrm{i}k}{2}, \frac{\mu+\mathrm{i}k}{2}, \mu; z\right). \tag{5.41}$$

We rewrite (3.23) as

$$\int_{-\infty}^0 \mathrm{d}z\, z^{\mu-1} F_k(z) P_{m-1}^{(M-N,0)}(z) = (-1)^\mu \frac{4\Gamma(\mu)^2}{k^2-k'^2}$$
$$\times\left[\mathrm{i}(k-k')\left(\frac{\Gamma(\mathrm{i}k)\Gamma(\mathrm{i}k')}{[\Gamma(\frac{\mu+\mathrm{i}k}{2})\Gamma(\frac{\mu+\mathrm{i}k'}{2})]^2}\right)(-z)^{\mathrm{i}(k+k')/2}\right.$$

$$-\mathrm{i}(k+k')\left(\frac{\Gamma(-\mathrm{i}k)\Gamma(\mathrm{i}k')}{[\Gamma(\frac{\mu-\mathrm{i}k}{2})\Gamma(\frac{\mu+\mathrm{i}k'}{2})]^2}\right)(-z)^{\mathrm{i}(-k+k')/2}$$
$$-\mathrm{i}(k-k')\left(\frac{\Gamma(-\mathrm{i}k)\Gamma(-\mathrm{i}k')}{[\Gamma(\frac{\mu-\mathrm{i}k}{2})\Gamma(\frac{\mu-\mathrm{i}k'}{2})]^2}\right)(-z)^{-\mathrm{i}(k+k')/2}$$
$$+\mathrm{i}(k+k')\left(\frac{\Gamma(\mathrm{i}k)\Gamma(-\mathrm{i}k')}{[\Gamma(\frac{\mu+\mathrm{i}k}{2})\Gamma(\frac{\mu-\mathrm{i}k'}{2})]^2}\right)(-z)^{-\mathrm{i}(-k+k')/2}\Bigg]_{-\infty}^{0} \tag{5.42}$$

and choose $k' = -\mathrm{i}(2m+M-N-1)$. Then $a_{mn}(t)$ is evaluated as follows

$$a_{mn}(t) = 4(-1)^{\mu}\Gamma(\mu)^2\int_{-\infty}^{\infty}\mathrm{d}k\,k^{2(n-1)}C(k)^2\exp[-\epsilon(k)t]\frac{\mathrm{i}}{k+k'}$$
$$\times\Bigg[\left(\frac{\Gamma(\mathrm{i}k)\Gamma(\mathrm{i}k')}{[\Gamma(\frac{\mu+\mathrm{i}k}{2})\Gamma(\frac{\mu+\mathrm{i}k'}{2})]^2}\right)(-z)^{\mathrm{i}(k+k')/2}$$
$$-\left(\frac{\Gamma(-\mathrm{i}k)\Gamma(-\mathrm{i}k')}{[\Gamma(\frac{\mu-\mathrm{i}k}{2})\Gamma(\frac{\mu-\mathrm{i}k'}{2})]^2}\right)(-z)^{-\mathrm{i}(k+k')/2}\Bigg]_{-\infty}^{0}$$
$$= \frac{1}{2\pi}(-1)^{\mu}\int_{-\infty}^{\infty}\mathrm{d}k\,k^{2(n-1)}\exp[-\epsilon(k)t]\frac{\mathrm{i}}{k+k'}$$
$$\times\Bigg[\left(\Gamma\left(\frac{\mu-\mathrm{i}k}{2}\right)^2\frac{\Gamma(\mathrm{i}k')}{\Gamma(-\mathrm{i}k)[\Gamma(\frac{\mu+\mathrm{i}k'}{2})]^2}\right)(-z)^{\mathrm{i}(k+k')/2}$$
$$-\left(\Gamma\left(\frac{\mu+\mathrm{i}k}{2}\right)^2\frac{\Gamma(-\mathrm{i}k')}{\Gamma(\mathrm{i}k)[\Gamma(\frac{\mu-\mathrm{i}k'}{2})]^2}\right)(-z)^{-\mathrm{i}(k+k')/2}\Bigg]_{-\infty}^{0}$$
$$= 2(-1)^{\mu-1}(-k')^{2(n-1)}\exp[-\epsilon(-k')t]$$
$$= 2(-1)^{\mu-1}\Big[-\{2(m-1)+\mu\}^2\Big]^{(n-1)}\exp[-\epsilon(\mathrm{i}\{2(m-1)+\mu\})t]. \tag{5.43}$$

The last equality follows from the contour integral. Next, we substitute $k^{2(j-1)}$ in the last expression of (5.41) by

$$L_j(k) = \prod_{l=1(l\neq j)}^{N}\frac{-k^2-(2(l-1)+\mu)^2}{(2(j-1)+\mu)^2-(2(l-1)+\mu)^2}. \tag{5.44}$$

This manipulation is justified because $k^{2(j-1)}$ appears in the determinant in (5.40), i.e. the the determinant does not change under this substitution besides a normalization constant. Note that

$$L_j(\mathrm{i}[2(m-1)+\mu]) = \delta_{j,m} \tag{5.45}$$

for $1 \leqslant j, m \leqslant N$. We see from (5.43) and (5.45) that

$$\int_{-\infty}^{0}\mathrm{d}z\,z^{M-N}P_{m-1}^{(M-N,0)}(z)\int_{0}^{\infty}\mathrm{d}k\,L_n(k)C(k)^2\exp[-\epsilon(k)t]F\left(\frac{\mu-\mathrm{i}k}{2},\frac{\mu+\mathrm{i}k}{2},\mu;z\right)$$
$$= \delta_{mn}2(-1)^{\mu-1}\exp[-\epsilon(-k')t] \tag{5.46}$$

for $1 \leqslant m, n \leqslant N$. Now let us set

$$K(z,z';t) = 2^{-1}(-1)^{1-\mu}\sum_{m=1}^{N}z^{M-N}P_{m-1}^{(M-N,0)}(z)s\times\int_{0}^{\infty}\mathrm{d}k\,L_m(k)C(k)^2$$
$$\times\exp\left[-\frac{a^2\{k^2+(2(m-1)+\mu)^2\}}{2}t\right]F\left(\frac{\mu-\mathrm{i}k}{2},\frac{\mu+\mathrm{i}k}{2},\mu;z'\right). \tag{5.47}$$

From (5.46) we obtain the relations

$$\int_{-\infty}^{0} \mathrm{d}z\, K(z, z) = N \tag{5.48}$$

$$\int_{-\infty}^{0} \mathrm{d}z'\, K(z, z'; t) K(z', z''; t) = K(z, z''; t) \tag{5.49}$$

and

$$\hat{G}(z, t) = \frac{1}{N!} \det[K(z_i, z_j; t)]_{i,j}. \tag{5.50}$$

The probability density function (5.50) is readily integrated over arbitrary numbers of points by using relations (5.48) and (5.49). This is the well known procedure in random matrix theory [16]. Consequently, we obtain the n-point correlation function exactly;

$$R_n(z_1, \ldots, z_n; t) = \det[K(z_i, z_j; t)]_{i,j=1,2,\ldots,n}. \tag{5.51}$$

6. Summary and discussions

In this paper we have studied a diffusion process on cosets $U(M+N)/U(M) \times U(N)$ and $U(M, N)/U(M) \times U(N)$ and their zero-curvature limit from a unified view point. In the zero-curvature limit we obtain a formula for an extension of the Itzykson–Zuber integral to rectangular matrices. By its correpsonding restricted root system we can classify the integral (1.3) for $M \neq N$ as BC-type, whereas (1.1) is A-type. The integral in [19] is obtained by setting $M = N$ in (1.3) and classified as C-type. This kind of extension is rather familiar in the study of integrable particle systems. It is fair to mention that the integral formula (4.24) has already been reported by Jackson *et al* [20]. There we find some mistakes concerning the decomposition of the symmetric space, which do not affect the final results. Let us add a comment on integral (1.3). In [8] Feinberg and Zee studied a rectangular matrix model. Using the technique of Hermitian reduction, they showed some interesting properties of rectangular random matrices. Among them is a modified partition function: for an $N \times M$ matrix X,

$$Z_A = \int \prod_{i=1}^{N} \prod_{j=1}^{M} \mathrm{Re}\,(\mathrm{d}X_{ij}) \mathrm{Im}\,(\mathrm{d}X_{ij}) \exp(-\sqrt{MN}\, \mathrm{tr}[XX^\dagger + XX^\dagger A]) \tag{6.1}$$

$A : N \times N$ Hermitian matrix.

This partition function can be analysed by the usual Itzykson–Zuber integral (1.1). While (6.1) and (1.1) are convenient from the view point of Hermitian reduction, a partition function

$$Z_B = \int \prod_{i=1}^{N} \prod_{j=1}^{M} \mathrm{Re}\,(\mathrm{d}X_{ij}) \mathrm{Im}\,(\mathrm{d}X_{ij}) \exp(-\sqrt{MN}\, \mathrm{tr}[XX^\dagger + XB]) \tag{6.2}$$

$B : M \times N$matrix

is also introduced naturally in consideration of the BC-type symmetry of the model. In order to analyse (6.2) the integral (1.3) must be used in place of (1.1). Thus, it is clear that the results in this paper should be useful for the random matrix theory with (6.2), or the probability density function,

$$\frac{1}{Z_B} \exp(-\sqrt{MN}\, \mathrm{tr}[XX^\dagger + XB]). \tag{6.3}$$

We might think of several possible applications of the diffusions studied in sections 2 and 3. First, the diffusion discussed in section 3 is applicable to quantum transport phenomena. Taking this purpose into consideration we have investigated in more detail the isotropic diffusion on $U(M,N)/U(M)\times U(N)$ in section 5. We have obtained the explicit form of the probability density function for the metallic and insulating regimes. The probability density function for the metallic regime is obtained via the integral representation of the hypergeometric function and is valid for arbitrary value of $\{x_i\}$. We have also obtained the n-point correlation function R_n exactly by using the method of orthogonal polynomials.

Appendix A. Normalization

For the discussions in section 5 the expression (3.26) for the normalization $C(k)$ is not convenient. Let us suppose first that $\mu = M-N+1$ is an odd integer and introduce a positive integer L by $M-N=2L$. Then using the double angle formula for Gamma functions,

$$\sqrt{\pi}\,\Gamma(2x) = 2^{2x-1}\Gamma(x)\Gamma(x+\tfrac{1}{2}) \tag{A.1}$$

and the known fact for Gamma functions at pure imaginary numbers,

$$|\Gamma(\mathrm{i}k)| = \left[\frac{\pi}{k\sinh\pi k}\right]^{1/2} \qquad k\in\mathbb{R} \tag{A.2}$$

the absolute value of $\Gamma(\frac{\mu}{2}+\frac{\mathrm{i}k}{2})$ is expressed as

$$\begin{aligned}\left|\Gamma\left(\frac{\mu}{2}+\frac{\mathrm{i}k}{2}\right)\right| &= \left|\Gamma\left(L+\frac{\mathrm{i}k}{2}+\frac{1}{2}\right)\right| \\ &= 2^{1-L}\sqrt{\pi}\left|\frac{\Gamma(\mathrm{i}k)}{\Gamma(\frac{\mathrm{i}k}{2})2^{\mathrm{i}k}}(\mathrm{i}k+1)(\mathrm{i}k+3)\cdots(\mathrm{i}k+2L-1)\right| \\ &= \frac{2^{\frac{1}{2}-L}\sqrt{\pi}}{\cosh^{1/2}(\pi k/2)}\left[\prod_{n=1}^{L}\{k^2+(2n-1)^2\}\right]^{1/2}.\end{aligned} \tag{A.3}$$

From (A.2) and (A.3), we obtain

$$\left|\frac{\Gamma(\frac{\mu}{2}+\frac{\mathrm{i}k}{2})^2}{\Gamma(\mathrm{i}k)}\right| = 2^{1-2L}[2\pi k\tanh(\pi k/2)]^{1/2}\prod_{n=1}^{L}[k^2+(2n-1)^2]. \tag{A.4}$$

When μ is an even integer we set $\mu=2L'$. It is easy to compute the absolute value of $\Gamma(\frac{\mu+\mathrm{i}k}{2})$;

$$\left|\Gamma\left(\frac{\mu+\mathrm{i}k}{2}\right)\right| = 2^{-L'}\left[\frac{2\pi}{k\sinh\pi k/2}\right]^{1/2}\prod_{n=1}^{L'}\left[k^2+4(n-1)^2\right]^{1/2}. \tag{A.5}$$

It follows that

$$\left|\frac{\Gamma(\frac{\mu}{2}+\frac{\mathrm{i}k}{2})^2}{\Gamma(\mathrm{i}k)}\right| = 2^{-2L'+3/2}\pi^{1/2}[k\tanh(\pi k/2)]^{-1/2}\prod_{n=1}^{L'}[k^2+4(n-1)^2]. \tag{A.6}$$

Using (A.4) and (A.6) in (3.26) we obtain explicit expressions for $C(k)$, which give (5.17) and (5.30).

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